

# REPRESENTATIONS OF $\mathfrak{asl}_2$

SOPHIE MORIER-GENOUD

**ABSTRACT.** We study representations of the simple Lie antialgebra  $\mathfrak{asl}_2$  introduced in [5]. We show that representations of  $\mathfrak{asl}_2$  are closely related to the famous ghost Casimir element of the universal enveloping algebra  $\mathfrak{osp}(1|2)$ . We prove that  $\mathfrak{asl}_2$  has no non-trivial finite-dimensional representations; we define and classify some particular infinite-dimensional representations that we call weighted representations.

## INTRODUCTION

“Lie antialgebras” is a new class of algebras introduced by V. Ovsienko [5]. These algebras naturally appear in the context of symplectic and contact geometry of  $\mathbb{Z}_2$ -graded spaces, their algebraic properties are not yet well understood. Lie antialgebras is a surprising “mixture” of commutative algebras and Lie algebras.

The first example of Ovsienko’s algebras is a simple Lie antialgebra  $\mathfrak{asl}_2(\mathbb{K})$ , over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . This algebra is of dimension 3, it has linear basis  $\{\varepsilon; a, b\}$  subject to the following relations:

$$(0.1) \quad \begin{aligned} \varepsilon \cdot \varepsilon &= \varepsilon, \\ \varepsilon \cdot a &= a \cdot \varepsilon = \frac{1}{2}a, & \varepsilon \cdot b &= b \cdot \varepsilon = \frac{1}{2}b, \\ a \cdot b &= -b \cdot a = \frac{1}{2}\varepsilon. \end{aligned}$$

The notion of representation of a Lie antialgebra was also defined in [5], and the problem of classification of representations of simple Lie antialgebras was formulated. In this paper, we study representations of  $\mathfrak{asl}_2(\mathbb{K})$ .

The Lie antialgebra  $\mathfrak{asl}_2(\mathbb{K})$  is closely related to the simple classical Lie superalgebra  $\mathfrak{osp}(1|2)$ . For instance,  $\mathfrak{osp}(1|2) = \text{Der}(\mathfrak{asl}_2(\mathbb{K}))$ . It was shown in [5], that every representation of  $\mathfrak{asl}_2(\mathbb{K})$  is naturally a representation of  $\mathfrak{osp}(1|2)$ . The first problem is thus to determine the corresponding class of  $\mathfrak{osp}(1|2)$ -representations. It turns out that this class can be characterized by so-called ghost Casimir element of the universal enveloping algebra  $\mathcal{U}(\mathfrak{osp}(1|2))$ . This element first appeared in [6], see also [1, 2, 4].

**Theorem 1.** *There is a one-to-one correspondence between representations of  $\mathfrak{asl}_2(\mathbb{K})$  and representations of  $\mathfrak{osp}(1|2)$  such that*

$$(0.2) \quad \Gamma^2 = \frac{1}{4} \text{Id},$$

where  $\Gamma$  is the action of the ghost Casimir element.

We will see that in the case of a  $\mathbb{Z}_2$ -graded irreducible representation  $V = V_0 \oplus V_1$  of  $\mathfrak{asl}_2(\mathbb{K})$ , one has a more complete information on the action of the ghost Casimir element, namely, the  $\mathbb{Z}_2$ -grading can be chosen in such a way that

$$\Gamma|_{V_0} = -\frac{1}{2} \text{Id}, \quad \Gamma|_{V_1} = \frac{1}{2} \text{Id}.$$

We believe that this relation with the ghost Casimir element provides a better understanding for the nature of  $\mathfrak{asl}_2(\mathbb{K})$  itself.

In the finite-dimensional case, we prove the following

**Theorem 2.** *The Lie antialgebra  $\text{asl}_2(\mathbb{K})$  has no non-trivial finite-dimensional representations.*

Let us emphasize that the algebra  $\text{asl}_2(\mathbb{K})$  was defined in [5] as analog of the classical simple Lie algebra  $\text{sl}_2(\mathbb{K})$ , but is also has a certain similarity with the 3-dimensional Heisenberg algebra  $\mathfrak{h}_1$ . The above result is similar to the classical result that  $\mathfrak{h}_1$  has no non-trivial irreducible representations.

In the infinite-dimensional case, we restrict our study to the case of *weighted representations*. A weighted representation is a representation containing an eigenvector for the action of the Cartan element  $H \in \text{osp}(1|2)$ . We classify the irreducible weighted representations of  $\text{asl}_2(\mathbb{K})$ . We introduce a family of weighted representations  $V(\ell)$ , for  $\ell \in \mathbb{K}$  (see Section 3.2 for the construction). Considering the set of parameters  $\mathcal{P} = [-1, 1]$  in the real case, or  $\mathcal{P} = [-1, 1] \cup \{\ell \in \mathbb{C} \mid -1 \leq \text{Re}(\ell) < 1\}$  in the complex case, we obtain the complete classification of irreducible weighted representations.

**Theorem 3.** *Any irreducible weighted representation is isomorphic to a  $V(\ell)$  for a unique  $\ell \in \mathcal{P}$ .*

The paper is organized as follows. In Section 1, we recall the general definitions of Lie antialgebras and their representations introduced in [5]. In Section 2, we obtain preliminary results about the representations of  $\text{asl}_2$ . A link with the Lie algebra  $\text{osp}(1|2)$  and with the Casimir elements is established. We complete the proof of Theorem 1 in subsection 2.3. In Section 3, we introduce the notion of weighted representations and give the construction of the family of irreducible weighted representations  $V(\ell)$ ,  $\ell \in \mathbb{K}$ . In Section 4, we formulate our results concerning the representations  $V(\ell)$  and complete the proofs of Theorem 2 and Theorem 3. In the end of the paper we discuss some general aspects of representation theory of  $\text{asl}_2(\mathbb{K})$ , such as the tensor product of two representations.

**Acknowledgements.** I am grateful to V. Ovsienko for the statement of the problem and enlightening discussions.

## 1. LIE ANTIALGEBRAS AND THEIR REPRESENTATIONS

Let us give the definition of a Lie antialgebra equivalent to the original definition of [5]. Throughout the paper the ground vector field is  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .

**Definition 1.1.** A Lie antialgebra is a  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ , equipped with a bilinear product  $\cdot$  satisfying the following conditions.

- (1) it is even:  $\mathfrak{a}_i \cdot \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$ ;
- (2) it is supercommutative, *i.e.*, for all homogeneous elements  $x, y \in \mathfrak{a}$ ,

$$x \cdot y = (-1)^{p(x)p(y)} y \cdot x$$

where  $p$  is the parity function defined by  $p(x) = i$  for  $x \in \mathfrak{a}_i$ ;

- (3) the subspace  $\mathfrak{a}_0$  is a commutative associative algebra;
- (4) for all  $x_1, x_2 \in \mathfrak{a}_0$  and  $y \in \mathfrak{a}_1$ , one has

$$x_1 \cdot (x_2 \cdot y) = \frac{1}{2} (x_1 \cdot x_2) \cdot y,$$

in other words, the subspace  $\mathfrak{a}_1$  is a module over  $\mathfrak{a}_0$ , homomorphism  $\varrho : \mathfrak{a}_0 \rightarrow \text{End}(\mathfrak{a}_1)$  being given by  $\varrho_x y = 2x \cdot y$  for all  $x \in \mathfrak{a}_0$  and  $y \in \mathfrak{a}_1$ ;

- (5) for all  $x \in \mathfrak{a}_0$  and  $y_1, y_2 \in \mathfrak{a}_1$ , the following Leibniz identity

$$x \cdot (y_1 \cdot y_2) = (x \cdot y_1) \cdot y_2 + y_1 \cdot (x \cdot y_2)$$

is satisfied;

- (6) for all  $y_1, y_2, y_3 \in \mathfrak{a}_1$ , the following Jacobi-type identity

$$y_1 \cdot (y_2 \cdot y_3) + y_2 \cdot (y_3 \cdot y_1) + y_3 \cdot (y_1 \cdot y_2) = 0$$

is satisfied.

**Example 1.2.** It is easy to see that the above axioms are satisfied for  $\mathfrak{asl}_2(\mathbb{K})$ . In this case, the element  $\varepsilon$  spans the even part,  $\mathfrak{asl}_2(\mathbb{K})_0$ , while the elements  $a, b$  span the odd part,  $\mathfrak{asl}_2(\mathbb{K})_1$ .

Consider a  $\mathbb{Z}_2$ -graded vector space  $V = V_0 \oplus V_1$ , the space  $\text{End}(V)$  of linear endomorphisms of  $V$  is a  $\mathbb{Z}_2$ -graded associative algebra:

$$\text{End}(V)_0 = \text{End}(V_0) \oplus \text{End}(V_1), \quad \text{End}(V)_1 = \text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0).$$

Following [5], we define the following “anticommutator” on  $\text{End}(V)$ :

$$(1.3) \quad ]X, Y[ := XY + (-1)^{p(X)p(Y)} YX,$$

where  $p$  is the parity function on  $\text{End}(V)$  and  $X, Y \in \text{End}(V)$  are homogeneous (purely even or purely odd) elements. Note that the sign rule in (1.3) is opposite to that of the usual commutator.

**Remark 1.3.** Let us stress that the operation (1.3) does *not* define a Lie antialgebra structure on the full space  $\text{End}(V)$  and it is not known what are the subspaces of  $\text{End}(V)$  for which this is the case. This operation provides, however, a definition of the notion of representation of a Lie antialgebra.

**Definition 1.4.** (a) We call *representation* of the Lie antialgebra  $\mathfrak{a}$ , the data of  $(V, \chi)$  where  $V = V_0 \oplus V_1$  is a  $\mathbb{Z}_2$ -graded vector space and  $\chi : \mathfrak{a} \rightarrow \text{End}(V)$  is an even linear map such that

$$(1.4) \quad ]\chi_x, \chi_y[ = \chi_{x \cdot y},$$

for all  $x, y \in \mathfrak{a}$ .

(b) A *subrepresentation* is a  $\mathbb{Z}_2$ -graded subspace  $V' \subset V$  stable under  $\chi_x$  for all  $x \in \mathfrak{a}$ .

(c) A representation is called *irreducible* if it does not have proper subrepresentations.

(d) Two representations  $(V, \chi)$  and  $(V', \chi')$  are called *equivalent* if there exists a linear map  $\Phi : V \rightarrow V'$  such that  $\Phi \circ \chi_x = \chi'_x \circ \Phi$ , for every  $x \in \mathfrak{a}$ .

**Remark 1.5.** It is not clear *a priori*, that a given Lie antialgebra has at least one non-trivial representation. In the case of  $\mathfrak{asl}_2(\mathbb{K})$ , however, an example of representation was given in [5] in the context of geometry of the supercircle.

Let us finally mention that there is a notion of module over a Lie antialgebra which is different from that of representation. For instance, the “adjoint action” defined as usual by  $\text{ad}_x y = x \cdot y$  is *not* a representation, but it defines a structure of  $\mathfrak{a}$ -module on  $\mathfrak{a}$ .

Given a Lie antialgebra  $\mathfrak{a}$ , it was shown in [5] that there exists a Lie superalgebra,  $\mathfrak{g}_{\mathfrak{a}}$ , canonically associated to  $\mathfrak{a}$ . Every representation of  $\mathfrak{a}$  extends to a representation of  $\mathfrak{g}_{\mathfrak{a}}$ . In the case of  $\mathfrak{asl}_2(\mathbb{K})$ , the corresponding Lie superalgebra is the classical

simple Lie antialgebra  $\mathfrak{osp}(1|2)$ . We will give the explicit construction of this Lie superalgebra in the next section and use it as the main tool for our study.

## 2. REPRESENTATIONS OF $\mathfrak{asl}_2(\mathbb{K})$ AND THE GHOST CASIMIR OF $\mathfrak{osp}(1|2)$

In this section we collect the general information about the representations of  $\mathfrak{asl}_2(\mathbb{K})$ . We also introduce the action of  $\mathfrak{osp}(1|2)$  and prove Theorem 1.

**2.1. Generators of  $\mathfrak{asl}_2(\mathbb{K})$  and the  $\mathbb{Z}_2$ -grading.** Consider an  $\mathfrak{asl}_2(\mathbb{K})$ -representation  $V = V_0 \oplus V_1$  with  $\chi : \mathfrak{asl}_2(\mathbb{K}) \rightarrow \text{End}(V)$ . The homomorphism condition (1.4) can be written explicitly in terms of the basis elements:

$$\begin{cases} \chi_a \chi_b - \chi_b \chi_a &= \frac{1}{2} \chi_\varepsilon \\ \chi_a \chi_\varepsilon + \chi_\varepsilon \chi_a &= \frac{1}{2} \chi_a \\ \chi_b \chi_\varepsilon + \chi_\varepsilon \chi_b &= \frac{1}{2} \chi_b \\ \chi_\varepsilon \chi_\varepsilon &= \frac{1}{2} \chi_\varepsilon. \end{cases}$$

Let us simplify the notations by fixing the following elements of  $\text{End}(V)$ :

$$A = 2 \chi_a, \quad B = 2 \chi_b, \quad \mathcal{E} = 2 \chi_\varepsilon.$$

The above relations read:

$$(2.5) \quad \begin{cases} AB - BA &= \mathcal{E} \\ A\mathcal{E} + \mathcal{E}A &= A \\ B\mathcal{E} + \mathcal{E}B &= B \\ \mathcal{E}^2 &= \mathcal{E}. \end{cases}$$

The element  $\mathcal{E}$  is a projector in  $V$ . This leads to a decomposition of  $V$  into eigenspaces  $V = V^{(0)} \oplus V^{(1)}$  defined by

$$V^{(\lambda)} = \{v \in V \mid \mathcal{E}v = \lambda v\}, \quad \lambda = 0, 1.$$

This decomposition is not necessarily the same as the initial one,  $V = V_0 \oplus V_1$ . Since  $V_i$ , where  $i = 0, 1$ , is stable under the action of  $\mathcal{E}$ , this gives a refinement:

$$V = V_0^{(0)} \oplus V_0^{(1)} \oplus V_1^{(0)} \oplus V_1^{(1)}$$

where

$$V_i^{(\lambda)} = \{v \in V_i \mid \mathcal{E}v = \lambda v\}, \quad \lambda = 0, 1, \quad i = 0, 1.$$

**Proposition 2.1.** *Any  $\mathfrak{asl}_2(\mathbb{K})$ -representation  $(V = V_0 \oplus V_1, \chi)$  is equivalent to a representation  $(V' = V'_0 \oplus V'_1, \chi')$  such that*

$$\mathcal{E}|_{V'_0} = 0, \quad \mathcal{E}|_{V'_1} = \text{Id}.$$

*Proof.* Using the relations (2.5) it is easy to see that  $A$  and  $B$  send the spaces  $V_i^{(\lambda)}$  into  $V_{1-i}^{(1-\lambda)}$ , where  $\lambda = 0, 1$ ,  $i = 0, 1$ . Thus, changing the  $\mathbb{Z}_2$ -grading of  $V$  to  $V' = V'_0 \oplus V'_1$  where

$$\begin{aligned} V'_0 &= V_0^{(0)} \oplus V_1^{(0)}, \\ V'_1 &= V_0^{(1)} \oplus V_1^{(1)}, \end{aligned}$$

does not change the parity of the operators  $A$ ,  $B$  and  $\mathcal{E}$  viewed as elements of the  $\mathbb{Z}_2$ -graded space  $\text{End}(V')$ . In other words, the map  $\chi' : \mathfrak{a} \rightarrow \text{End}(V')$  defined by  $\chi'_x = \chi_x$  for all  $x \in \mathfrak{a}$ , is still an even map satisfying the condition (1.4).

By consequent,  $(V', \chi')$  is also a representation. It is then clear that  $(V', \chi')$  is equivalent (in the sense of Definition 1.4 (d)) to  $(V, \chi)$ .  $\square$

As a consequence of the above proposition we will always assume in the sequel that the  $\mathbb{Z}_2$ -grading of the representations  $V$  is given by the eigenspaces of the action  $\mathcal{E}$ .

**2.2. Action of  $\mathfrak{osp}(1|2)$ .** This section provides a special case of the general construction of [5], For the sake of completeness, we give here the details of the computations.

Given an  $\mathfrak{asl}_2(\mathbb{K})$ -representation  $V$ , we define the operators  $E$ ,  $F$  and  $H$  by

$$(2.6) \quad E = A^2, \quad F = -B^2, \quad H = -(AB + BA).$$

These three elements define a structure of  $\mathfrak{sl}_2(\mathbb{K})$ -module on  $V$ .

**Lemma 2.2.** *One has:*

$$(2.7) \quad \begin{aligned} [H, E] &= 2E \\ [H, F] &= -2F \\ [E, F] &= H \end{aligned}$$

*Proof.* These relations follow from relations (2.5). Indeed,

$$\begin{aligned} [H, E] &= -(AB + BA)A^2 + A^2(AB + BA) \\ &= -ABA^2 - BA^3 + A^3B + A^2BA \\ &= -2ABA^2 + (ABA^2 - BA^3) + (A^3B - A^2BA) + 2A^2BA \\ &= -2ABA^2 + (AB - BA)A^2 + A^2(AB - BA) + 2A^2BA \\ &= 2A(AB - BA)A + (AB - BA)A^2 + A^2(AB - BA) \\ &= 2A\mathcal{E}A + \mathcal{E}A^2 + A^2\mathcal{E} \\ &= (A\mathcal{E}A + \mathcal{E}A^2) + (A\mathcal{E}A + A^2\mathcal{E}) \\ &= (A\mathcal{E} + \mathcal{E}A)A + A(\mathcal{E}A + A\mathcal{E}) \\ &= A^2 + A^2 \\ &= 2E. \end{aligned}$$

In the same way we obtain  $[H, F] = -2F$ . Finally,

$$\begin{aligned} [E, F] &= -A^2B^2 + B^2A^2 \\ &= -A^2B^2 + ABAB - ABAB + BA^2B - BA^2B + BABA - BABA + B^2A^2 \\ &= -A(AB - BA)B - (AB - BA)AB - BA(AB - BA) - B(AB - BA)A \\ &= -A\mathcal{E}B - \mathcal{E}AB - BA\mathcal{E} - B\mathcal{E}A \\ &= -(A\mathcal{E} + \mathcal{E}A)B - B(A\mathcal{E} + \mathcal{E}A) \\ &= -AB - BA \\ &= H \end{aligned}$$

$\square$

With similar computations one can establish the following additional relations:

$$(2.8) \quad \begin{array}{lll} [H, A] & = A & [E, A] = 0 \quad [F, A] = B, \\ [H, B] & = -B & [E, B] = A \quad [F, B] = 0, \\ [H, \mathcal{E}] & = 0 & [E, \mathcal{E}] = 0 \quad [F, \mathcal{E}] = 0, \end{array}$$

that can be summarized as follows.

**Proposition 2.3.** *Every representation of  $\mathfrak{asl}_2(\mathbb{K})$  has a structure of a module over the Lie superalgebra  $\mathfrak{osp}(1|2)$ . The even part,  $\mathfrak{osp}(2|1)_0$ , is spanned by  $E, F, H$ , while the odd part,  $\mathfrak{osp}(2|1)_1$ , is spanned by  $A, B$ .*

The relation (2.7) and (2.8) play an essential role in all our computations.

**2.3. The ghost Casimir element.** The notion of *twisted adjoint action* was introduced for a certain class of Lie superalgebras in [2]. We recall here the definition in the  $\mathfrak{osp}(1|2)$ -case.

Let  $X$  be an element of  $\mathfrak{osp}(1|2)$  and  $Y$  be an element of the universal enveloping algebra  $\mathcal{U}(\mathfrak{osp}(1|2))$ . Define  $\widetilde{\text{ad}} : \mathfrak{osp}(1|2) \rightarrow \text{End}(\mathcal{U}(\mathfrak{osp}(1|2)))$  by

$$(2.9) \quad \widetilde{\text{ad}}_X Y := XY - (-1)^{p(X)(p(Y)+1)} YX.$$

In other words,

$$\widetilde{\text{ad}}_X = \begin{cases} \text{ad}_X & \text{if } X \text{ is even} \\ -\text{ad}_X & \text{if } X \text{ is odd.} \end{cases}$$

Remarkably enough,  $\widetilde{\text{ad}}$  defines an  $\mathfrak{osp}(1|2)$ -action on  $\mathcal{U}(\mathfrak{osp}(1|2))$ .

The *ghost Casimir* elements are the invariants of the twisted adjoint action, see [2], and also [4]. In the case of  $\mathfrak{osp}(1|2)$ , the ghost Casimir element is particularly simple:

$$(2.10) \quad \Gamma = AB - BA - \frac{1}{2} \text{Id},$$

The ghost Casimir satisfies  $\widetilde{\text{ad}}_X \Gamma = 0$  for all  $X \in \mathfrak{osp}(1|2)$ .

The relation between the above twisted adjoint action and our situation is the following. Consider a representation  $\chi : \mathfrak{asl}_2(\mathbb{K}) \rightarrow \text{End}(V)$ . Denote by  $U$  the subalgebra of  $\text{End}(V)$  generated by the image of  $\mathfrak{asl}_2(\mathbb{K})$  under  $\chi$ . The algebra  $U$  can be viewed as a quotient of  $\mathcal{U}(\mathfrak{osp}(1|2))$ . A simple comparison of (2.9) and (1.3) shows that, if  $x$  is an odd element of  $\mathfrak{asl}_2(\mathbb{K})$ , then

$$]\chi_x, Y[ = \widetilde{\text{ad}}_{\chi_x}(Y),$$

for all  $Y$  in  $U$ . The operator  $\mathcal{E}$  and the ghost Casimir  $\Gamma$  are obviously related by

$$\Gamma = \mathcal{E} - \frac{1}{2} \text{Id}.$$

It follows that the second and third relations in (2.5) are equivalent to  $\widetilde{\text{ad}}_A \Gamma = 0$  and  $\widetilde{\text{ad}}_B \Gamma = 0$ , respectively, while the relation  $\mathcal{E}^2 = \mathcal{E}$  reads  $\Gamma^2 = \frac{1}{4} \text{Id}$ .

This completes the proof of Theorem 1.

**2.4. Usual Casimir elements.** The operator  $\mathcal{E}$  is also related to the usual Casimir elements  $C$ , resp.  $C_0$  of  $\mathfrak{osp}(1|2)$ , resp.  $\mathfrak{sl}_2(\mathbb{K})$ . Recall

$$\begin{aligned} C &= EF + FE + \frac{1}{2}(H^2 + AB - BA), \\ C_0 &= EF + FE + \frac{1}{2}H^2. \end{aligned}$$

We easily see

$$\mathcal{E} = 2(C - C_0).$$

This implies that, if  $V$  is an irreducible representation of  $\mathfrak{asl}_2(\mathbb{K})$ , then  $\mathcal{E}|_{V_0}$  and  $\mathcal{E}|_{V_1}$  are proportional to  $\text{Id}$ .

Moreover, straightforward computation in  $\mathcal{U}(\mathfrak{osp}(1|2))$  gives the following relation

$$4(C - C_0)^2 = 4C - 2C_0.$$

It follows the condition  $\mathcal{E}^2 = \mathcal{E}$  implies that  $C$  acts trivially.

### 3. WEIGHTED REPRESENTATIONS OF $\mathfrak{asl}_2(\mathbb{K})$ .

In this section, we introduce the notion of weighted representation of the Lie antialgebra  $\mathfrak{asl}_2(\mathbb{K})$ . This class of representation is characterized by the property that the action of the Cartan element  $H$  of  $\mathfrak{osp}(1|2)$  has at least one eigenvector. We do not require *a priori* the eigenspaces to be finite dimensional.

**3.1. The definition.** Let  $V$  be a representation of  $\mathfrak{asl}_2(\mathbb{K})$ . We introduce the subspaces

$$V_\ell = \{v \in V \mid Hv = \ell v\}, \ell \in \mathbb{K}.$$

Whenever  $V_\ell \neq \{0\}$ , we call this subspace a *weight space* of  $V$  with weight  $\ell$ . We denote by  $\Pi_H(V)$  the set of weights of representation  $V$ .

**Lemma 3.1.** *With the above notations:*

- (i) *The element  $A$  (resp.  $B$ ) maps  $V_\ell$  into  $V_{\ell+1}$  (resp.  $V_{\ell-1}$ ).*
- (ii) *The sum  $\sum_{\ell \in \Pi_H(V)} V_\ell$  is direct in  $V$ .*
- (iii) *The space*

$$Wt(V) := \bigoplus_{\ell \in \Pi_H(V)} V_\ell$$

*is a subrepresentation of  $V$ .*

*Proof.* Let  $v$  be a vector in  $V_\ell$ . Using the relations (2.8) we obtain:

$$HA v = [H, A]v + AHv = Av + \ell Av = (\ell + 1)v$$

and

$$HBv = [H, B]v + BHv = -Bv + \ell Bv = (\ell - 1)v.$$

Part (i) then follows.

Part (ii) is clear since the weight spaces are eigenspaces for  $H$ .

It follows from (i) that  $Wt(V)$  is stable with respect to the action of  $A$  and  $B$  and, therefore, it is also stable under  $\mathcal{E} = AB - BA$ . Hence (iii).  $\square$

**Corollary 3.2.** *If  $V$  is an irreducible representation then either*

$$Wt(V) = \{0\} \quad \text{or} \quad Wt(V) = V.$$

**Definition 3.3.** We call *weighted representation* any representation  $V$  of  $\mathfrak{asl}_2(\mathbb{K})$  such that  $Wt(V) \neq \{0\}$ .

**3.2. The family of weighted representations  $V(\ell)$ .** For every  $\ell \in \mathbb{K}$ , we construct an irreducible weighted representation of  $\mathfrak{asl}_2(\mathbb{K})$  that we denote  $V(\ell)$ . This representation contains an odd vector  $e_1$  such that  $He_1 = \ell e_1$  and, by irreducibility, every element of  $V(\ell)$  is a result of the (iterated)  $\mathfrak{asl}_2(\mathbb{K})$ -action on  $e_1$ .

(a) **The case where  $\ell$  is not an odd integer.** We start the construction with the generic weight  $\ell$ .

Consider a family of linearly independent vectors  $\{e_k\}_{k \in \mathbb{Z}}$ . We set  $V(\ell) = \bigoplus_{k \in \mathbb{Z}} \mathbb{K}e_k$  and we define the operators  $A$  and  $B$  on  $V(\ell)$  by

$$\begin{aligned} Ae_k &= e_{k+1}, \quad \forall k \in \mathbb{Z} \\ Be_k &= ((1-\ell)/2 - [k/2])e_{k-1}, \quad \forall k \in \mathbb{Z}, \end{aligned}$$

The operator  $\mathcal{E}$  is determined by  $\mathcal{E} = AB - BA$ . Introduce the following  $\mathbb{Z}_2$ -grading on  $V(\ell)$ :

$$\begin{aligned} V(\ell)_0 &= \bigoplus_{k \text{ even}} \mathbb{K}e_k, \\ V(\ell)_1 &= \bigoplus_{k \text{ odd}} \mathbb{K}e_k. \end{aligned} \tag{3.11}$$

It is easy to see the operators  $A$  and  $B$  are odd operators with respect to this grading whereas  $\mathcal{E}$  is even.

**Proposition 3.4.** *The space  $V(\ell)$  together with the operators  $A, B, \mathcal{E}$  is an  $\mathfrak{asl}_2(\mathbb{K})$ -representation.*

*Proof.* By simple straightforward computations we obtain:

$$A\mathcal{E} + \mathcal{E}A = A, \quad B\mathcal{E} + \mathcal{E}B = B.$$

Moreover, on the basis elements  $e_k$  of  $V(\ell)$

$$\mathcal{E}e_k = \begin{cases} e_k, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}$$

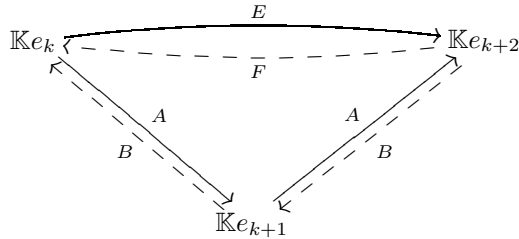
so that  $\mathcal{E}^2 = \mathcal{E}$ . □

It is easy to see the basis elements  $e_k$ 's are weight vectors. Indeed, one checks

$$He_k = (\ell + k - 1)e_k, \quad \forall k \in \mathbb{Z}. \tag{3.12}$$

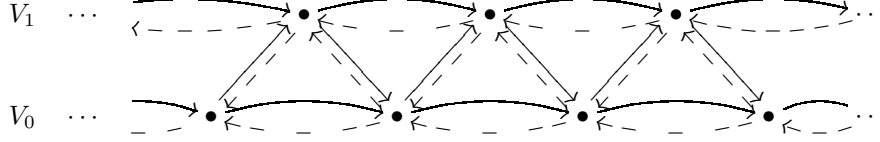
In particular, the element  $e_1$  is a weight vector of weight  $\ell$  and generates the representation  $V(\ell)$ .

The actions on the basis elements of  $\mathfrak{osp}(1|2)$  can be pictured as follows:





The entire space  $V(\ell)$  can be pictured as a infinite chain of the above diagrams.



(b) **Construction of  $V(\ell)$  for  $\ell$  a positive odd integer.** Consider a family of linearly independent vectors  $\{e_k\}_{k \in \mathbb{Z}, k \geq 2-\ell}$ . We set  $V(\ell) = \bigoplus_{k \geq 2-\ell} \mathbb{K}e_k$  and we define the operators  $A$  and  $B$  on  $V(\ell)$  by a similar formula:

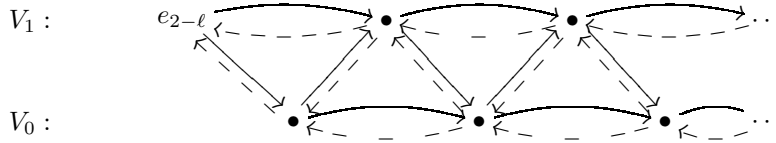
$$\begin{aligned} Ae_k &= e_{k+1}, \quad \forall k \geq 2-\ell \\ Be_k &= ((1-\ell)/2 - [k/2])e_{k-1}, \quad \forall k > 2-\ell, \\ Be_{2-\ell} &= 0. \end{aligned}$$

The operator  $\mathcal{E}$  is again determined by  $\mathcal{E} = AB - BA$ . The  $\mathbb{Z}_2$ -grading on  $V(\ell)$  is defined by the same formula (3.11). The result of Lemma 3.4 holds true.

The element  $e_1$  is an odd weight vector of weight  $\ell$  and generates the representation  $V(\ell)$ . However, the vector  $e_{2-\ell}$  is more interesting.

**Definition 3.5.** We call a representation  $V$  a lowest weight (resp. highest weight) representation if it contains a vector  $v$ , such that  $Bv = 0$  and the vectors  $A^n v$  span  $V$  (resp.  $Av = 0$  and  $B^n v$  span  $V$ ); the vector  $v$  is called a lowest weight (resp. highest weight) vector.

Clearly, the vector  $e_{2-\ell}$  is a lowest weight vector of the representation  $V(\ell)$ , if  $\ell$  is a positive odd integer. One obtains the following diagram.

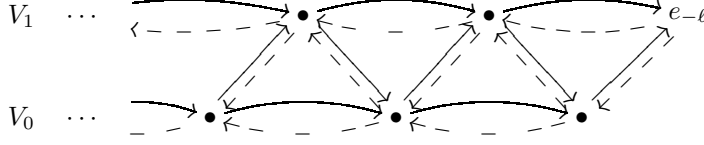


Viewed as a representation of  $\mathfrak{osp}(1|2)$ ,  $V(\ell)$  is a Verma module.

(c) **Construction of  $V(\ell)$  for  $\ell$  a negative odd integer.** Consider a family of linearly independent vectors  $\{e_k\}_{k \in \mathbb{Z}, k \leq -\ell}$ . We set  $V(\ell) = \bigoplus_{k \leq -\ell} \mathbb{K}e_k$  and we define the operators  $A$  and  $B$  on  $V(\ell)$  by

$$\begin{aligned} Ae_k &= ((1+\ell)/2 + [k/2])e_{k+1}, \quad \forall k < -\ell, \\ Ae_{-\ell} &= 0, \\ Be_k &= e_{k-1}, \quad \forall k \leq -\ell. \end{aligned}$$

As previously these operators define an  $\mathfrak{asl}_2(\mathbb{K})$ -representation. The vector  $e_{-\ell}$  is a highest weight vector of  $V(\ell)$ .



**3.3. Geometric realization.** It was shown in [5] that  $\mathfrak{asl}_2(\mathbb{K})$  has a representation in terms of vector fields on the 1|1-dimensional space. More precisely, consider  $\mathcal{F} = C_{\mathbb{K}}^{\infty}(\mathbb{R})$  the set of  $\mathbb{K}$ -valued  $C^{\infty}$ -functions of one real variable  $x$ . Introduce  $\mathcal{A} = \mathcal{F}[\xi]/(\xi^2)$  with the  $\mathbb{Z}_2$ -grading  $\mathcal{A}_0 = \mathcal{F}$ ,  $\mathcal{A}_1 = \mathcal{F}\xi$ . Define the vector field

$$\mathcal{D} = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}.$$

It is very easy to check that the following vector fields:

$$(3.13) \quad A = \mathcal{D}, \quad B = x\mathcal{D}, \quad \mathcal{E} = \xi\mathcal{D}$$

satisfy the relations (2.5). Therefore this defines an  $\mathfrak{asl}_2(\mathbb{K})$ -action on  $\mathcal{A}$ .

Let us choose the following function:

$$e_1 = x^{\lambda} \xi,$$

where  $\lambda \in \mathbb{K}$ . It turns out this function generates a weighted representation of  $\mathfrak{asl}_2(\mathbb{K})$ , isomorphic to  $V(\ell)$ , with the weight  $\ell = -2\lambda - 1$ . Note that the case  $\lambda$  is an integer gives the highest or lowest irreducible representation.

**Remark 3.6.** The vector field  $\mathcal{D}$  defines the standard contact structure on  $\mathbb{K}^{1|1}$ . The vector fields (3.13) are therefore *tangent* to the contact structure. These vector fields do not span a Lie (super)algebra with respect to the usual Lie bracket. Remarkable enough, the corresponding generators of the  $\mathfrak{osp}(1|2)$ -action:

$$E = \frac{\partial}{\partial x}, \quad F = -x^2 \frac{\partial}{\partial x} - \xi \mathcal{D}, \quad H = -2x \frac{\partial}{\partial x} - \xi \mathcal{D}$$

are contact vector fields, while the above vector fields  $A$  and  $B$  are the only vector fields that are contact and tangent at the same time. The Lie superalgebra  $\mathfrak{osp}(1|2)$  thus preserves the contact structure in the usual way.

#### 4. CLASSIFICATION RESULTS

In this section we prove Theorem 3 which is our main classification result.

**4.1. Classification of weighted representations.** The following statement shows that the representations  $V(\ell)$  are, indeed, irreducible; it also classifies all the isomorphisms between these representations.

**Proposition 4.1.** (i) For every  $\ell \in \mathbb{K}$ , the representation  $V(\ell)$  is an irreducible infinite-dimensional representation.

(ii) Two weighted representations  $V(\ell)$  and  $V(\ell')$ , where  $\ell$  and  $\ell'$  are not odd integers, are isomorphic if and only if  $\ell' - \ell = 2m$  for some  $m \in \mathbb{Z}$ .

(iii) Two weighted representations  $V(\ell)$  and  $V(\ell')$ , where  $\ell$  and  $\ell'$  are odd integers, are isomorphic if and only if  $\ell$  and  $\ell'$  have same sign.

*Proof.* (i) Irreducibility of  $V(\ell)$ :

Suppose there exists a subrepresentation  $V'$  of  $V(\ell)$ . Consider a nonzero vector  $v \in V'$  and write

$$v = \sum_{1 \leq i \leq N} \alpha_i e_{k_i}$$

with  $\alpha_i \neq 0$ , for all  $1 \leq i \leq N$ . Using (3.12), we obtain,

$$H^p v = \sum_{1 \leq i \leq N} \alpha_i (\ell + k_i - 1)^p e_{k_i},$$

where  $0 \leq p \leq N - 1$ . We may assume the coefficients  $(\ell + k_i - 1)$  are nonzero. In other words, we assume the basis elements occurring in the decomposition of  $v$  are not of weight zero. If this is not the case, we change  $v$  to  $A^k v$  or  $B^k v$  for sufficiently large  $k$ .

The above equations form a linear system of type Vandermonde with distinct nonzero coefficients, and so it is solvable. We can express the  $e_{k_i}$ 's as linear combinations of  $H^p v$ 's. It follows that the vectors  $e_{k_i}$  are in  $V'$ . Applying  $A$  and  $B$  to  $e_{k_i}$ , we produce all the vectors  $e_k$ ,  $k \in \mathbb{Z}$ . Hence, the vectors  $e_k$ , of the basis are in  $V'$  for all  $k \in \mathbb{Z}$ . This implies  $V' = V(\ell)$ . Therefore there is no proper subrepresentation of  $V(\ell)$ .

(ii) Let  $\ell$  and  $\ell'$  be two scalars in  $\mathbb{K}$  that are not odd integers.

Denote by  $\{e_k\}_k$  the standard basis of  $V(\ell)$  and  $\{e'_k\}_k$  the standard basis of  $V(\ell')$ . Suppose there exists an isomorphism of representation  $\Phi : V(\ell') \rightarrow V(\ell)$ .

The vector  $\Phi(e'_1)$  is a vector of weight  $\ell'$  in  $V(\ell)$ . The weights in  $V(\ell)$  are of the form  $\ell + k$  for some  $k \in \mathbb{Z}$  and the corresponding weight space is  $\mathbb{K}e_{k+1}$ . We thus have  $\Phi(e'_1) = \alpha e_{k+1}$  for some  $\alpha \in \mathbb{K} \setminus \{0\}$ ,  $k \in \mathbb{Z}$ , and therefore  $\ell' = \ell + k$ .

Moreover,  $\mathcal{E}\Phi(e'_1) = \Phi(e'_1)$ , so  $e_{k+1}$  has to be an odd vector, *i.e.*,  $k$  has to be even.

We proved that a necessary condition to have  $V(\ell)$  isomorphic to  $V(\ell')$  is

$$\ell' = \ell + 2m,$$

where  $m \in \mathbb{Z}$ .

Conversely, suppose  $\ell' = \ell + 2m$ , for some  $m \in \mathbb{Z}$ . It is easy to check that the linear map  $\Phi : V(\ell') \rightarrow V(\ell)$  defined by

$$\Phi(e'_k) = e_{k+2m},$$

for all  $k \in \mathbb{Z}$ , is an isomorphism of representation.

(iii) Let  $\ell$  and  $\ell'$  be two odd integers. If  $\ell$  and  $\ell'$  have opposite sign then  $V(\ell)$  and  $V(\ell')$  cannot be isomorphic, since on one of the space  $A$  acts injectively and on the other space  $A$  has a non-trivial kernel.

Conversely, if  $\ell$  and  $\ell'$  have same sign, let us construct an explicit isomorphism between  $V(\ell)$  and  $V(\ell')$ . One has:  $\ell' = \ell - 2m$ , for some  $m \in \mathbb{N}$  and we define  $\Phi : V(\ell') \rightarrow V(\ell)$  by

$$\Phi(e'_k) = e_{k+2m}, \quad \forall k \geq 2 - \ell'$$

in the case where  $\ell$  and  $\ell'$  are positive, or by

$$\Phi(e'_k) = e_{k+2m}, \quad \forall k \leq -\ell'$$

in the case where  $\ell$  and  $\ell'$  are negative. □

**4.2. Structure of weighted representations.** In this section we establish two lemmas crucial for the proofs of Theorem 2 and 3.

Let us consider an irreducible representation  $V$  of  $\text{asl}_2(\mathbb{K})$ . Recall that the  $\mathbb{Z}_2$ -grading  $V = V_0 \oplus V_1$  corresponds to the decomposition with respect to the eigenvalues of  $\mathcal{E}$ , see Lemma 2.1.

**Lemma 4.2.** *If there exists a nonzero element  $v \in V$  such that  $Hv = \ell v$ ,  $\ell \in \mathbb{K}$ , then there exist nonzero elements  $v_i \in V_i$  and  $\ell_i \in \mathbb{K}$ ,  $i = 0, 1$ , such that  $Hv_i = \ell_i v_i$ .*

*Proof.* In the case where  $v$  is a pure homogeneous element, i.e.,  $v \in V_i$ , we can choose for  $v_{1-i}$  the vector  $Av$  or  $Bv$ . Indeed, one has

$$(4.14) \quad \begin{aligned} HAv &= [H, A]v + AHv = Av + \ell Av = (\ell + 1)v, \\ HBv &= [H, B]v + BHv = -Bv + \ell Av = (\ell - 1)v. \end{aligned}$$

If the two vectors  $Av$  and  $Bv$  are zero then  $V = \mathbb{K}v$  is the trivial representation and, by consequent, the statement of the theorem is true. Otherwise, if  $Av$  and  $Bv$  are not zero, one obtains weight vectors of weight  $\ell_{1-i} = \ell \pm 1$ .

If  $v$  is not a homogeneous element, we can write  $v = v_0 + v_1$  with  $v_0 \neq 0 \in V_0$  and  $v_1 \neq 0 \in V_1$ . One then has:

$$Hv = Hv_0 + Hv_1 \text{ and } Hv = \ell v = \ell v_0 + \ell v_1.$$

Furthermore,  $Hv_0$  is an element of  $V_0$  and  $Hv_1$  is an element of  $V_1$ , since the operator  $H = -(AB + BA)$  is even. Therefore, by uniqueness of the writing in  $V_0 \oplus V_1$ , one has:

$$Hv_0 = \ell v_0 \text{ and } Hv_1 = \ell v_1.$$

□

**Lemma 4.3.** *If  $v \in V_i$  is such that  $Hv = \ell v$  then :*

(i) *for all  $k \geq 1$ ,*

$$AB^k v = \lambda_k B^{k-1} v$$

*with  $\lambda_k = \left\lfloor \frac{k-i}{2} \right\rfloor + \frac{i-\ell}{2}$ , where  $\left\lfloor \frac{k-i}{2} \right\rfloor$  is the integral part of  $(k-i)/2$ ;*

(ii) *for all  $k \geq 1$ ,*

$$BA^k v = \mu_k A^{k-1} v$$

*with  $\mu_k = -\left\lfloor \frac{k-i}{2} \right\rfloor - \frac{i+\ell}{2}$ .*

*Proof.* We first establish the formulas for  $k = 1$ . On the one hand one has:

$$ABv = -Hv - BAv = -\ell v - BAv$$

and on the other hand,

$$ABv = \mathcal{E}v + BAv = iv + BAv.$$

By adding or subtracting these two identities we deduce:

$$2ABv = (i - \ell)v \text{ and } 2BAv = (-i - \ell)v.$$

We get  $\lambda_1 = (i - \ell)/2$  and  $\mu_1 = (-i - \ell)/2$ , so that the formulas are established at the order 1.

By induction on  $k$ ,

$$\begin{aligned} ABB^{k-1}v &= -HB^{k-1}v - BAB^{k-1}v \\ &= -(\ell - (k-1))B^k v - \lambda_{k-1}B^{k-1}v \\ &= (-\ell + k - 1 - \lambda_{k-1})B^{k-1}v \end{aligned}$$

We deduce the relations:

$$\begin{aligned} \lambda_k &= -\ell + k - 1 - \lambda_{k-1} \\ &= -\ell + k - 1 - (-\ell + (k-2) - \lambda_{k-2}) \\ &= 1 + \lambda_{k-2}. \end{aligned}$$

Knowing  $\lambda_1 = \frac{i-\ell}{2}$  we can now obtain the explicit expression of  $\lambda_k$ :

$$\lambda_k = \left\lfloor \frac{k-i}{2} \right\rfloor + \frac{i-\ell}{2}.$$

Hence part (i).

Part (ii) can be proved in a similar way.  $\square$

**4.3. Proof of Theorem 2.** In this section we prove that  $\mathfrak{asl}_2(\mathbb{K})$  has no non-trivial representations of finite dimension.

Let  $V$  be an irreducible finite dimensional representation of  $\mathfrak{asl}_2(\mathbb{K})$ . Considering the actions of the elements  $E = A^2$ ,  $F = -B^2$  and  $H$ , the space  $V$  has a structure of  $\mathfrak{sl}_2$ -module. Therefore, there exists a weight vector  $v$  such that  $Hv = \ell v$  for some  $\ell \in \mathbb{Z}$ .

By lemma 4.2, we can assume that  $v$  is a homogeneous element, namely  $v \in V_i$ ,  $i = 0, 1$ . Let us consider the family of vectors

$$\mathcal{F} = \{\dots, B^k v, \dots, Bv, v, Av, \dots, A^k v, \dots\}.$$

From formula (4.14) we know that all the nonzero vectors of  $\mathcal{F}$  are eigenvectors of  $H$  with distinct eigenvalues,  $\ell \pm k$ ,  $k \in \mathbb{N}$ . Therefore, all the non-zero vectors of  $\mathcal{F}$  are linearly independent.

Hence, there exists  $N \geq 1$  such that

$$B^{N-1}v \neq 0, \quad B^k v = 0, \quad \forall k \geq N$$

and  $M \geq 1$  such that

$$A^{M-1}v \neq 0, \quad A^k v = 0, \quad \forall k \geq M.$$

Using Lemma 4.3 we deduce

$$\begin{aligned} \lambda_N &= 0, \\ \mu_M &= 0. \end{aligned}$$

This leads to

$$\begin{aligned} \left\lfloor \frac{N-i}{2} \right\rfloor + \frac{i-\ell}{2} &= 0, \\ -\left\lfloor \frac{M-i}{2} \right\rfloor - \frac{i+\ell}{2} &= 0. \end{aligned}$$

By subtracting these two equations we obtain:

$$\left\lfloor \frac{N-i}{2} \right\rfloor + \left\lfloor \frac{M-i}{2} \right\rfloor + i = 0.$$

But one has:  $N, M \geq 1$  and  $i = 0, 1$ . So, necessarily,

$$N = M = 1, \quad i = 0.$$

In conclusion,  $v$  is an even vector such that  $Av = Bv = 0$  and  $V$  is nothing but the trivial representation.

Finally, if  $V$  is an arbitrary finite-dimensional representation, then  $V$  is completely reducible. This immediately follows from the classical theorem in the  $\mathfrak{osp}(1|2)$ -case.

Theorem 2 is proved.

**4.4. Proof of Theorem 3.** Let us consider an infinite-dimensional irreducible weighted representation  $V$ .

We start by studying the cases of highest weight representations and lowest weight representations.

**Lemma 4.4.** *Every irreducible highest weight representation is isomorphic to  $V(-1)$ .*

*Proof.* Consider an irreducible representation  $V$  containing a weight vector  $v$  of weight  $\ell$  such that  $Av = 0$ . We write  $v = v_0 + v_1$  with  $v_i \in V_i$ ,  $i = 0, 1$ . We also have  $Av_i = 0$  and  $Hv_i = \ell v_i$  for  $i = 0, 1$ .

We first show that  $v_0 = 0$ . Consider the action of  $\mathfrak{sl}_2(\mathbb{K})$  on  $V_0$ , the vector  $v_0$  is a highest weight vector for this action. Since  $Av_0 = 0$ , we get

$$Hv_0 = ABv_0 = -\mathcal{E}v_0 = 0.$$

The vector  $v_0$  is a highest weight vector of weight 0 for the action of  $\mathfrak{sl}_2(\mathbb{K})$ . By consequence,  $v_0$  is also a lowest weight vector, *i.e.*,  $B^2v_0 = 0$ . Thus, the space  $\text{Span}(v_0, Bv_0)$  is stable under the action of  $A$  and  $B$ . Since  $V$  is an infinite-dimensional irreducible representation one necessarily has  $v_0 = 0$ .

We can assume now that  $v$  belongs to  $V_1$ . Let us use Lemma 4.3, part (ii). From the relation  $BAv = 0$ , we deduce  $\mu_1 = 0$  and thus  $\ell = -1$ . This implies that all the constants  $\lambda_k$  from Lemma 4.3, part (i) are non-zero. By induction we deduce, using Lemma 4.3, part (i), that all the vectors  $B^k v$ ,  $k \in \mathbb{N}$  are non-zero. Moreover, these vectors are linearly independent since they are eigenvectors for  $H$  associated to distinct eigenvalues. By setting

$$e_k = B^{1-k}v, \quad k \in \mathbb{Z}, \quad k \leq 1,$$

we obtain  $V(-1)$  as a subrepresentation of  $V$ . We then deduce from the irreducibility assumption that  $V \simeq V(-1)$ .  $\square$

**Lemma 4.5.** *Every irreducible lowest weight representation is isomorphic to  $V(1)$ .*

*Proof.* Similar to the proof of Lemma 4.4.  $\square$

Now we are ready to prove Theorem 3.

Fix a weight vector  $v \in V_1$  (such a vector exists by Lemma 4.2) of some weight  $\ell \in \mathbb{K}$ . Consider the family

$$\mathcal{F} := \{A^k v, B^k v, k \in \mathbb{N}\}.$$

(a) Suppose that  $\ell$  is not an odd integer. It is easy to see that the constants  $\lambda_k$  and  $\mu_k$ ,  $k \in \mathbb{N}$ , from Lemma 4.3 never vanish. Indeed,

$$(4.15) \quad \begin{aligned} \lambda_k = 0 &\Rightarrow \ell = 2\left[\frac{k-1}{2}\right] + 1, \\ \mu_k = 0 &\Rightarrow \ell = -2\left[\frac{k-1}{2}\right] - 1. \end{aligned}$$

By induction, we deduce that the elements in  $\mathcal{F}$  are different from zero. Moreover, the elements of  $\mathcal{F}$  are eigenvectors for the operator  $H$  with distinct eigenvalue  $\ell \pm k$ , where  $k \in \mathbb{N}$ , so that, they are linearly independent.

By setting

$$e_k = \begin{cases} A^{k-1}v, & k \geq 1 \\ B^{1-k}v, & k \leq 0 \end{cases}$$

we see that  $V(\ell)$  as a subrepresentation of  $V$ . Again, by irreducibility assumption, we deduce  $V \simeq V(\ell)$ .

Finally, using Proposition 4.1, part (ii), one has:

$$V \simeq V(\ell'),$$

where  $\ell'$  is the unique element of

$$\mathcal{P}^+ = [-1, 1] \cup \{\ell \in \mathbb{C} \mid -1 \leq \operatorname{Re}(\ell) < 1\},$$

such that  $\ell' = \ell + 2m$  for some  $m \in \mathbb{Z}$ .

(b) Suppose that  $\ell$  is a positive odd integer. From the first statement of (4.15), we deduce the existence of an integer  $N \geq 1$ , such that  $\lambda_N = 0$  and  $\lambda_k \neq 0$  for all  $k < N$ . Hence,

$$B^k v \neq 0, \quad \forall k < N, \quad AB^N v = 0.$$

If  $B^N v \neq 0$  then this vector is a highest weight vector. By Lemma 4.4, we obtain  $V \simeq V(-1)$ . But, in the highest weight representation  $V(-1)$ , the set of weights is the set of negative integers. We obtain a contradiction since  $v$  has a positive weight.

It follows that  $B^N v = 0$  and this implies that  $B^{N-1}$  is a lowest weight vector. Using Lemma 4.5, we conclude

$$V \simeq V(1).$$

(c) Suppose finally that  $\ell$  is a negative odd integer. Then similar arguments show:

$$V \simeq V(-1).$$

Theorem 3 is proved.

**Remark 4.6.** We proved that any irreducible weighted representation is a Harish-Chandra irreducible representation (*i.e.* the weight spaces are all finite dimensional). A classification of Harish-Chandra irreducible representations of  $\mathfrak{osp}(1|2)$  over the field of complex numbers is given in [3]. The correspondence between the representations  $V(\ell)$  and the representations given in Theorem 5.13 of [3] is the following:

(a) If  $\ell \in \mathcal{P}^+$  is not an odd integer then

$$V(\ell) \simeq \mathcal{D}(l, \lambda_0),$$

for the choice  $l = 0$  and  $\lambda_0 = \ell/2$ .

(b) The lowest weight representation is

$$V(1) \simeq [\lambda_0] \downarrow,$$

for the unique choice  $\lambda_0 = -1/2$ .

(c) The highest weight representation is

$$V(-1) \simeq [\lambda_0] \uparrow,$$

for the unique choice  $\lambda_0 = 1/2$ .

#### APPENDIX: TENSOR PRODUCT OF TWO REPRESENTATIONS

Given two representations,  $V$  and  $W$  of  $\mathfrak{asl}_2(\mathbb{K})$ , to what extent their tensor product,  $V \otimes W$  is again an  $\mathfrak{asl}_2$ -representation? This question is non-trivial since  $\mathfrak{asl}_2(\mathbb{K})$  is not a Lie algebra. We will show that  $\mathfrak{asl}_2(\mathbb{K})$  does not act on  $V \otimes W$ . An attempt to define such an action leads to a deformation of the  $\mathfrak{asl}_2$ -relations by the Casimir element of  $\mathcal{U}(\mathfrak{osp}(1|2))$ . The algebraic meaning of this deformation is not yet clear.

The operators  $A$  and  $B$  have canonical lifts to  $V \otimes W$  according to the Leibniz rule:

$$\tilde{A} = A \otimes \text{Id} + \text{Id} \otimes A, \quad \tilde{B} = B \otimes \text{Id} + \text{Id} \otimes B,$$

since they belong to the  $\mathfrak{osp}(1|2)$ -action. It is then natural to define the lift of operator  $\mathcal{E}$  by  $\tilde{\mathcal{E}} := \tilde{A}\tilde{B} - \tilde{B}\tilde{A}$ . One immediately obtains the explicit formula

$$\tilde{\mathcal{E}} = \mathcal{E} \otimes \text{Id} + \text{Id} \otimes \mathcal{E} + 2(A \otimes B - B \otimes A).$$

The following statement is straightforward.

**Proposition 4.7.** *The operators  $\tilde{A}, \tilde{B}$  and  $\tilde{\mathcal{E}}$  satisfy the following relations:*

$$\begin{aligned} \tilde{A}\tilde{B} - \tilde{B}\tilde{A} &= \tilde{\mathcal{E}} \\ \tilde{A}\tilde{\mathcal{E}} + \tilde{\mathcal{E}}\tilde{A} &= \tilde{A} \\ \tilde{B}\tilde{\mathcal{E}} + \tilde{\mathcal{E}}\tilde{B} &= \tilde{B} \\ \tilde{\mathcal{E}}^2 &= \tilde{\mathcal{E}} + 4\bar{C}, \end{aligned} \tag{4.16}$$

where

$$\bar{C} = E \otimes F + F \otimes E + \frac{1}{2}(H \otimes H + A \otimes B - B \otimes A).$$

This means that two of the relations (2.5) are satisfied, but not the last  $\mathfrak{asl}_2$ -relation  $\mathcal{E}^2 = \mathcal{E}$ .

Let us recall that the element  $C \in \mathcal{U}(\mathfrak{osp}(1|2))$  given by

$$C = EF + FE + \frac{1}{2}(H^2 + AB - BA)$$

is nothing but the classical Casimir element. The operator  $\bar{C}$  is the diagonal part of the standard lift of  $C$  to  $V \otimes W$ . In particular, the operator  $\bar{C}$  commutes with the action of  $\mathfrak{asl}_2(\mathbb{K})$  and  $\mathfrak{osp}(1|2)$ :

$$[\bar{C}, \tilde{A}] = [\bar{C}, \tilde{B}] = [\bar{C}, \tilde{\mathcal{E}}] = 0.$$

This is how the Casimir operator of  $\mathfrak{osp}(1|2)$  appears in the context of representations of  $\mathfrak{asl}_2(\mathbb{K})$ .

The relations (4.16) look like a “deformation” of the  $\mathfrak{asl}_2$ -relations (2.5) with one parameter that commutes with all the generators. It would be interesting to find a precise algebraic sense of this deformation.



## REFERENCES

- [1] D. Arnal, H. Ben Amor, G. Pinczon, *The structure of  $sl(2,1)$  supersymmetry*, Pacific J. of Math., 165,1,1994,17-49.
- [2] D. Arnaudon, M. Bauer, L. Frappat, *On Casimir's ghost*, Comm. Math. Phys. **187:2** (1997), 429-439.
- [3] H. Benamor, G. Pinczon, *Extensions of representations of Lie superalgebras*, J. Math. Phys. **32:3** (1991), 621-629.
- [4] M. Gorelik, *On the ghost centre of Lie superalgebras*, Ann. Inst. Fourier **50:6** (2000), 1745-1764.
- [5] V. Ovsienko, *Lie antialgebras*, arXiv:0705.1629.
- [6] G. Pinczon, *The enveloping algebra of the Lie superalgebra  $osp(1,2)$* , J. of Algebra, 132,1990,1,219-242.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, USA  
E-mail address: `sophiemg@umich.edu`